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Geometry of Solutions of Hitchin Equations on \mathbb{R}^2

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Abstract

We study smooth $SU(2)$ solutions of the Hitchin equations on \mathbb{R}^2 , with the determinant of the complex Higgs field being a polynomial of degree n . When $n \geq 3$, there are moduli spaces of solutions, in the sense that the natural L^2 metric is well-defined on a subset of the parameter space. We examine rotationally-symmetric solutions for $n = 1$ and $n = 2$, and then focus on the $n = 3$ case, elucidating the moduli and describing the asymptotic geometry as well as the geometry of two totally-geodesic surfaces.

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1 Introduction

The dimensional reduction of the self-dual Yang-Mills equations to \mathbb{R}^2 has long been of interest, even though this system does not admit smooth finite-energy solutions with compact gauge group [1, 2]. But there are solutions if one allows singularities and/or infinite energy [3, 2], or for complex or non-compact gauge group [4, 5]. The equations are conformally invariant, and so one can also study the system on general Riemann surfaces; in that context, the equations are known as the Hitchin equations, and in particular there are smooth solutions on compact Riemann surfaces of genus at least two [6]. The \mathbb{R}^2 case, with appropriate boundary conditions, may be viewed as a system on S^2 with a singularity at infinity; and more generally one could allow several designated singularities. There is a natural L^2 metric on the space of solutions, although the integral defining it may not always converge; these moduli-space metrics, when well-defined, are hyperkähler [6, 7]. Such Hitchin systems and their moduli spaces have close connections with supersymmetric field theory: see, for example, reference [8].

The focus in this paper is on smooth $SU(2)$ solutions of the Hitchin equations on \mathbb{R}^2 . There is a natural boundary condition which involves a positive integer n : namely, the determinant H of the complex Higgs field Φ is taken to be a polynomial of degree n in $z = x + iy$. Then the solutions generically resemble n lumps on \mathbb{R}^2 , located at the zeros of H : the solutions are parametrized by these locations, together with relative angles between the lumps. Some of these parameters have L^2 variation, and these are moduli; this is only possible when $n \geq 3$, and the main aim below is to describe some of the geometry of the $n = 3$ moduli space \mathcal{M} . In particular, we calculate the asymptotic geometry by using a singular approximation to the fields, and obtain the metric on two natural totally-geodesic surfaces in \mathcal{M} with the help of a numerical calculation. To begin with, however, we recall the rotationally-symmetric $n = 1$ solution, and describe a one-parameter family of rotationally-symmetric $n = 2$ solutions.

Some of the methods used below first arose in studies of the Hitchin equations on the cylinder $\mathbb{R} \times S^1$, in connection with periodic monopoles [9, 10, 11, 12, 13]. Periodic monopoles correspond, via the generalized Nahm transform, to solutions of the Hitchin equations on the cylinder $\mathbb{R} \times S^1$ satisfying appropriate boundary conditions. Since the cylinder is conformally equivalent to the punctured plane, these Hitchin fields are defined on \mathbb{R}^2 minus a point (or S^2 minus two points), and so they do not overlap with the solutions considered below.

2 The Hitchin equations on \mathbb{R}^2

Let A_j be a smooth $SU(2)$ gauge potential on \mathbb{R}^2 , with corresponding gauge field $F := F_{xy} = \partial_x A_y - \partial_y A_x + [A_x, A_y]$, and let Φ denote a Higgs field taking values in the complexification of the Lie algebra $\mathfrak{su}(2)$. The Hitchin equations [1, 6] are

$$D_{\bar{z}}\Phi = 0, \quad F = \frac{i}{2} [\Phi, \Phi^*]. \quad (1)$$

Here $z = x + iy$ is a complex coordinate on \mathbb{R}^2 , $D_{\bar{z}}\Phi := \partial_{\bar{z}}\Phi + [A_{\bar{z}}, \Phi]$ is the covariant derivative, and Φ^* denotes the complex-conjugate transpose of Φ . The system (1) is completely-integrable in the sense of having a Lax pair, namely the reduction of the Lax pair for the self-dual Yang-Mills equations [14]. In addition, for example, the moduli spaces of solutions on Riemann surfaces may be regarded as completely-integrable Hamiltonian systems [6].

It follows from (1) that the determinant $\det \Phi$ is holomorphic in z ; as a boundary condition, we require that $H(z) = \det \Phi$ should be a polynomial in z , and that $F \rightarrow 0$ as $|z| \rightarrow \infty$. Such solutions generically have F peaked at the zeros of $H(z)$, and we may visualize these as lumps.

Suppose now that we have a family of solutions depending on a parameter t . Let $\dot{\Phi}$ and $\dot{A}_{\bar{z}}$ denote the t -derivatives of Φ and $A_{\bar{z}}$, representing a vector V on the space of solutions. If we impose a condition that this variation V be orthogonal to the gauge orbit at $(\Phi, A_{\bar{z}})$, then the combined equations satisfied by $(\dot{\Phi}, \dot{A}_{\bar{z}})$ are

$$D_{\bar{z}}\dot{\Phi} = [\Phi, \dot{A}_{\bar{z}}], \quad 4D_{\bar{z}}\dot{A}_{\bar{z}} = [\Phi, \dot{\Phi}^*]. \quad (2)$$

Given that $(\dot{\Phi}, \dot{A}_{\bar{z}})$ satisfy (2), the natural L^2 norm of V is

$$\|V\|^2 = \frac{1}{2} \int \text{tr} \left(\dot{\Phi} \dot{\Phi}^* + 4 \dot{A}_{\bar{z}} \dot{A}_{\bar{z}}^* \right) |dz|^2. \quad (3)$$

Although (3) is not the only metric one can define on a solution space or moduli space, it is the most natural one both from the geometrical point of view [6] and in relation to the dynamics of topological solitons [15]. If (3) converges, then t is a modulus; whereas if (3) diverges, we refer to t as a parameter. The moduli space \mathcal{M} is the space of moduli, for fixed values of the parameters.

A useful choice of gauge was described in [12, 16]; it may be summarized as follows. The gauge field F is taken to be in the σ_3 -direction in $\mathfrak{su}(2)$, and Φ has the form

$$\Phi = \begin{pmatrix} 0 & \mu_+ e^{\psi/2} \\ \mu_- e^{-\psi/2} & 0 \end{pmatrix}. \quad (4)$$

Here μ_{\pm} are polynomials in z satisfying $\mu_+\mu_- = -H$, and ψ is a smooth real-valued function. The gauge potential has the form

$$A_{\bar{z}} = -\frac{1}{4}(\partial_{\bar{z}}\psi)\sigma_3 + \alpha\Phi, \quad (5)$$

where α is smooth complex-valued function. The Hitchin equations then reduce to equations on ψ and α , namely

$$\Delta\psi = 2(1 + 4|\alpha|^2)(|\mu_+|^2 e^{\psi} - |\mu_-|^2 e^{-\psi}), \quad (6)$$

$$0 = e^{-\psi/2}\partial_z(e^{\psi}\mu_+\alpha) + e^{\psi/2}\partial_{\bar{z}}(e^{-\psi}\bar{\mu}_+\bar{\alpha}), \quad (7)$$

where $\Delta = \partial_x^2 + \partial_y^2 = 4\partial_z\partial_{\bar{z}}$. The residual gauge freedom consists of constant diagonal $SU(2)$ transformations, plus conjugating by σ_1 . In the special case $\alpha = 0$, the equation (6) is locally equivalent to the sinh-Gordon equation, via a conformal mapping combined with a transformation $\psi \mapsto \psi + f(z) + \overline{f(z)}$. In other words, the explicit z -dependence in (6) could then be removed, at the cost of more complicated global conditions on the field ψ .

3 Symmetries and the $n = 1$ solution

We are taking $H(z) = \det \Phi$ to be a polynomial of degree n , so $H(z) = p_0 z^n + \dots + p_n$, where the p_k are complex constants. The Hitchin equations are conformally invariant, but the quantity $|p_0|$ (say) sets the length scale, and from now on we shall fix this scale by setting $|p_0| = 1$. The equations also have a rotational symmetry $z \mapsto e^{i\chi}z$, as well as a global phase symmetry $\Phi \mapsto e^{i\phi}\Phi$, and these are in effect linked by the argument of p_0 ; we fix this phase by setting $p_0 = 1$.

Some solutions are rotationally-symmetric, which clearly can only happen if $H(z) = z^n$. We say that the field $(\Phi, A_{\bar{z}})$ with $\det \Phi = z^n$ is rotationally-symmetric if the rotated field

$$\tilde{\Phi}(z) = e^{-in\chi/2}\Phi(e^{i\chi}z), \quad \tilde{A}_{\bar{z}}(z) = e^{-i\chi}A_{\bar{z}}(e^{i\chi}z) \quad (8)$$

is gauge-equivalent to $(\Phi(z), A_{\bar{z}}(z))$. Rotationally-symmetric solutions were studied a long time ago, in effect only for the case $n = 0$ [3, 2]; and also more recently, where it was shown that in certain cases the solutions can be expressed in terms of Painlevé transcendents [17, 16].

Although most of the global phase symmetry has been removed by fixing the phase of p_0 as we did above, there is a remnant left over, namely $\Gamma : \Phi \mapsto -\Phi$. Now Γ changes the sign of the function α appearing in the gauge potential (5), so the Γ -invariant fields are precisely those with $\alpha = 0$. By putting $\chi = 2\pi$ in (8), we see that a rotationally-symmetric

field with n odd is also Γ -invariant. But for n even, there are rotationally-symmetric fields which are not Γ -invariant, as we shall see in the next section.

For the remainder of this section, let us focus on the $n = 1$ case. We can use translation freedom to set $H(z) = z$. There is a rotationally-symmetric $n = 1$ solution which has its gauge field F peaked at $z = 0$; it is given by (4) and (5) with $\mu_+ = z$, $\mu_- = -1$, $\psi = \psi(r)$ and $\alpha = 0$, where $r = |z|$. If we change variables to $t = r^{3/2}$ and $h(t) = e^{-\psi/2}t^{-1/3}$, then the remaining field equation (6) becomes a particular case of the Painlevé-III equation, namely

$$h'' - \frac{(h')^2}{h} + \frac{h'}{t} + \frac{4}{9h} - \frac{4h^3}{9} = 0. \quad (9)$$

Figure 1 shows the magnitude $|F(r)|$ of the corresponding gauge field, obtained by solving (9) numerically. We see that the gauge field is peaked at $z = 0$, and has a radius of order unity; it is in effect abelian, and so we can use Green's theorem to compute $\int |F| dx dy = \pi/2$.

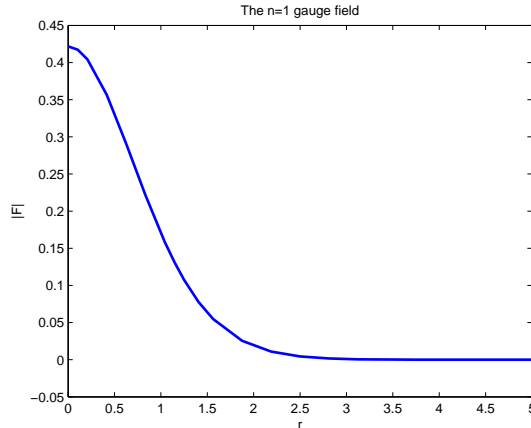


Figure 1: The profile function $w(r)$ and the gauge field $|F(r)|$ for the $n = 1$ solution.

Outside the core region around $z = 0$, there is an approximation to the field, or ‘limiting configuration’ [18, 19], which turns out to be a useful one: namely

$$\Phi = \sqrt{z} i\sigma_1, \quad A_x = 0, \quad A_y = 0 \text{ for } x > 0, \quad A_y = \frac{\pi}{2} \delta(y) i\sigma_3 \text{ for } x < 0. \quad (10)$$

Here the square root is branched along the negative x -axis, where the gauge potential has a delta-function singularity. The corresponding gauge field F is supported at $z = 0$: in fact, it is a 2-dimensional delta-function $F = -(\pi/2) \delta^2(x, y) i\sigma_3$. The branch discontinuity is gauge-removable, but the singularity at $z = 0$ is not. For a given Φ as in (10), the gauge

field could instead be proportional to $\cos(\omega)\sigma_2 + \sin(\omega)\sigma_3$, so there is an angle ω associated with the lump. In the $n = 1$ case, this angle is gauge-removable; but for $n > 1$ the relative phase between lumps plays a role, as shall see below.

4 The $n = 2$ case

The general form of $H(z)$ is $H(z) = z^n + p_1 z^{n-1} + \dots + p_n$, and the singular approximation generalizing (10), namely $\Phi = \sqrt{H} i\sigma_1$, can be used to determine which of the coefficients p_k are moduli. The convergence of (3) is determined by the asymptotic behaviour of the fields as $r \rightarrow \infty$, and the singular approximation is a good one in this asymptotic region. The L^2 norm of the derivative $\partial\Phi/\partial p_k$, for the singular field, is

$$\left\| \frac{\partial\Phi}{\partial p_k} \right\|^2 = \int_{\mathbb{R}^2} \frac{|z|^{2n-2k}}{4|H|} dx dy. \quad (11)$$

Assuming that the zeros of H are simple, this integral converges if and only if k lies in the range $(n+3)/2 \leq k \leq n$. So for these values of k , the coefficient p_k is a complex modulus. In particular, the $n = 1$ and $n = 2$ cases do not admit moduli, whereas for $n \geq 3$ we do get moduli.

In addition to the coefficients p_k , there are other moduli: these are the relative phases between the n lumps. If n is odd, then $(n-1)/2$ of the coefficients p_k are complex moduli, which together with $n-1$ relative phases gives a total of $2(n-1)$ real moduli. If n is even, then $(n-2)/2$ of the coefficients are complex moduli. In this case, there are $n-1$ relative phases, but one combination of them does not have L^2 variation, so we end up with a total of $2(n-2)$ real moduli in the even case. In particular, the $n = 3$ and $n = 4$ cases each have a 4-dimensional moduli space \mathcal{M} .

The Γ -invariant fields are exactly those for which the lumps are parallel or antiparallel, in other words the relative phases are 0 or π . This may be understood in the gauge (4, 5), as follows. Let $\{z_1, \dots, z_n\}$ denote the roots of $H(z)$. Then the relative phase between the lump at z_j and the one at z_k is the angle in $\mathfrak{su}(2)$ between the gauge fields $F^{(j)}$ and $F^{(k)}$ at these two points. To make sense of this, we have to specify how to compare the isovectors F at the two points. But for Γ -invariant fields we have $\alpha = 0$ in (5), and then parallel-propagation between the two points does not change F at all; so we conclude that $F^{(j)}$ and $F^{(k)}$, both being in the σ_3 -direction, are either parallel or antiparallel.

We now examine some features of the $n = 2$ case. Let \mathcal{S} denote the set of solutions which are invariant under Γ , up to gauge. Then \mathcal{S} has two components, namely \mathcal{S}_+ where $F^{(1)}$ and $F^{(2)}$ are parallel, and \mathcal{S}_- where they are antiparallel. Let \mathcal{R} denote the space of

rotationally-symmetric fields. Then $\mathcal{S}_- \cap \mathcal{R}$ has the two gauge fields cancelling: indeed, it is the explicit degenerate solution

$$\Phi = zi\sigma_1, \quad A_{\bar{z}} = 0. \quad (12)$$

The set \mathcal{R} of rotationally-symmetric solutions is a 1-parameter family of fields interpolating between (12) and the non-explicit field $\mathcal{S}_+ \cap \mathcal{R}$. This family can be obtained as solutions of a boundary-value problem for a single real-valued function, as follows.

We work in the gauge (4, 5), with $\mu_+ = z^2$ and $\mu_- = -1$. In the $n = 2$ case, the rotational-symmetry condition becomes $\psi = \psi(r)$ and $\alpha = \alpha(r)$. Then the general solution of equation (7) is $\alpha = B_+/M_+ + iB_-/M_-$, where B_{\pm} are real constants and

$$M_{\pm} = r^2 e^{\psi/2} \mp e^{-\psi/2}.$$

Now the boundary condition $F \rightarrow 0$ as $r \rightarrow \infty$ implies that $B_+ = 0$, so we end up with an expression for α in terms of $\psi(r)$ and the real constant $B = B_-$, namely $\alpha = iB/M_-$. This can then be substituted into (6) to give an equation for $\psi(r)$, namely

$$\psi'' + r^{-1}\psi' = 2[1 + 4B^2(r^2 e^{\psi/2} + e^{-\psi/2})^{-2}](r^4 e^{\psi} - e^{-\psi}). \quad (13)$$

The boundary conditions on ψ are $\psi'(0) = 0$ and $\psi(r) \sim -2 \log(r)$ as $r \rightarrow \infty$. In the $B = 0$ case (where $\alpha = 0$), it is again a Painlevé-III equation, but with a different parameter. Its nature for general B might be worth investigating, for example via a Painlevé analysis.

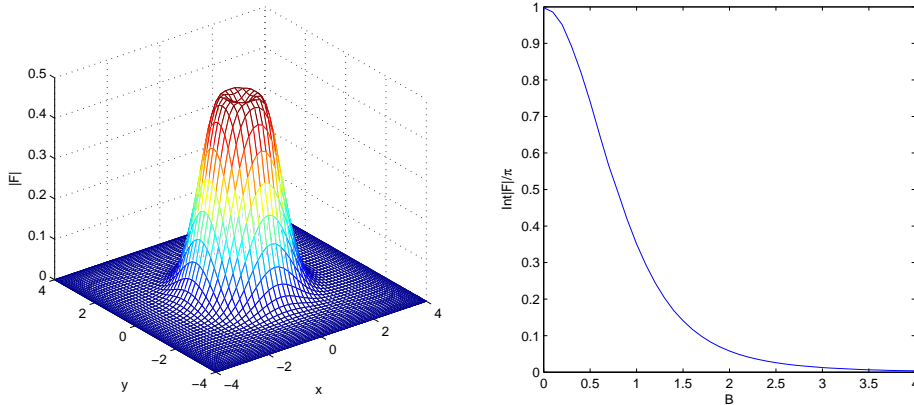


Figure 2: The gauge field $|F|$ for a rotationally-symmetric $n = 2$ field with $B = 0$; and $\pi^{-1} \int |F| d^2x$ as a function of B .

For each value of $B \geq 0$, we get a rotationally-symmetric solution, and this family is illustrated in Figure 2. The left-hand plot is the $B = 0$ case, and shows the norm $|F|$ of the gauge field as a function of x and y ; this was obtained by solving (6) numerically with $\alpha = 0$. The field $|F|$ is peaked on a circle centred at $z = 0$, and $\int |F| d^2x = \pi$. The right-hand diagram shows the result of solving the boundary-value problem (13) numerically, and plots $\pi^{-1} \int |F| d^2x$ as a function of B . The limit $B \rightarrow \infty$ is the degenerate solution (12) where $F = 0$, whereas $B = 0$ is the solution depicted in the left-hand plot.

5 The $n = 3$ case

If $n = 3$, then $H(z)$ has the form $H(z) = z^3 + p_1 z^2 + p_2 z + p_3$. The constants p_1 and p_2 are complex parameters, whereas p_3 is a complex modulus. By translation of z , we may set $p_1 = 0$. We can also use a rotation to make p_2 real. This leaves us with

$$H(z) = z^3 + az - K, \quad (14)$$

where $a > 0$ is a fixed parameter, and K is a complex modulus. For each value of a , we have a 4-dimensional moduli space \mathcal{M}_a , on which the local coordinates are the real and imaginary parts of K together with two relative phases η_1 and η_2 . Note that \mathcal{M}_0 admits a rotational Killing vector, since $K \mapsto e^{i\nu} K$ is then an isometry.

The asymptotic region of \mathcal{M}_a is where $|K| \gg 1$, and in this region the singular approximation $\Phi = \sqrt{H} i \sigma_1$ is a good one [18, 19]. The reason for this is that the lumps become smaller as they separate, *ie.* as $K \rightarrow \infty$. For example, if $H(z) = z^3 - b^3$, then $H(z) \approx 3b^2(z - b)$ near $z = b$, and so the lump at $z = b$ has characteristic size $1/(3b^2)$, and in effect it approaches a delta-function as $b \rightarrow \infty$. So the singular approximation is an accurate one in the asymptotic region of \mathcal{M} where the zeros of $H(z)$ are well-separated.

Let $\{z_1, z_2, z_3\}$ denote the zeros of (14). The gauge field near z_j has the approximate form

$$F^{(j)} = \frac{i\pi}{2} [\cos(\omega_j) \sigma_2 + \sin(\omega_j) \sigma_3] \delta(z - z_j).$$

This gives us three angles $\{\omega_1, \omega_2, \omega_3\}$ in the $\sigma_2 \sigma_3$ -plane, and hence two relative angles, say $\eta_1 = \omega_1 - \omega_3$ and $\eta_2 = \omega_2 - \omega_3$. The asymptotic moduli are K , η_1 and η_2 , and the structure is that of a T^2 -bundle over the K -space, with $\eta_1, \eta_2 \in [-\pi, \pi]$ being the T^2 coordinates. This bundle is twisted: if the phase of K goes from 0 to 2π , thereby permuting the zeros of $H(z)$, then the angles (η_1, η_2) are transformed by an element Υ of the modular group $\text{SL}(2, \mathbb{Z})$, namely

$$\Upsilon : (\eta_1, \eta_2) \mapsto (-\eta_2, \eta_1 - \eta_2).$$

In fact, Υ is a cube root of unity in $\text{SL}(2, \mathbb{Z})$.

One may now calculate the asymptotic metric on the moduli space from the integral formula (3), by using the singular approximation. The calculation is analogous to that described in [13], and will only be sketched here. For large $|K|$, the parameter a may be ignored, and so the singular approximation is $\Phi = \sqrt{z^3 - K} i\sigma_1$. Differentiating this with respect to K gives a vector V_1 , and its norm is $\|V_1\|^2 = \int |d\sqrt{z^3 - K}/dK|^2 |dz|^2$. The structure of the perturbation equations (2) enables us to write down three other vectors $\{V_2, V_3, V_4\}$, giving an orthogonal tetrad $\{V_\mu\}$ of vectors all having the same norm. For each of the V_μ , we can compute the corresponding change in the coordinates K, η_1, η_2 . This then gives us a coordinate expression for the metric, which turns out to be

$$ds^2 = c|K|^{-1/3}|dK|^2 + \frac{1}{\sqrt{3}}(d\eta_1^2 + d\eta_2^2 - d\eta_1 d\eta_2), \quad (15)$$

where

$$c = \frac{3\sqrt{3}}{4} \left(\int_0^1 \frac{du}{\sqrt{1-u^3}} \right)^2 \approx 2.554.$$

This metric is flat, with a conical singularity at $K = 0$. It is of type ALG [10], since the 4-volume of a ball with geodesic radius $R = |K|^{5/6}$ is proportional to R^2 for large R . It is invariant under the modular transformation Υ , and therefore well-defined.

Now $\Gamma : \mathcal{M}_a \rightarrow \mathcal{M}_a$ is an isometry, and so the subset $\mathcal{S} \subset \mathcal{M}_a$ of Γ -invariant fields is a 2-dimensional totally-geodesic submanifold of \mathcal{M}_a . In what follows, we study the geometry of this surface \mathcal{S} . Now \mathcal{S} has two components: one of them is \mathcal{S}_+ , where the three lumps are parallel ($\eta_1 = \eta_2 = 0$); and the other is \mathcal{S}_- , where two lumps are parallel and the third is antiparallel to them. Let us first consider \mathcal{S}_+ , which is diffeomorphic to \mathbb{R}^2 and has K as a global complex coordinate. The induced metric on \mathcal{S}_+ has the form $ds^2 = \Omega(K, \bar{K}) |dK|^2$. From (15) we expect that $\Omega \sim c|K|^{-1/3}$ as $|K| \rightarrow \infty$. In fact, $ds^2 = c|K|^{-1/3}|dK|^2$ is a cone, and so \mathcal{S}_+ is this cone with its vertex smoothed.

A numerical calculation can be used to obtain information about the geometry of \mathcal{S}_+ in its central region. The procedure is as follows: solve (6) with $\alpha = 0$ numerically, for a range of values of K ; obtain tangent vectors $V = (\dot{\Phi}, \dot{A}_{\bar{z}})$ by evaluating the difference in the fields for neighbouring values of K and then projecting orthogonal to the gauge orbits; evaluate the integral (3) to calculate Ω ; and finally compute the Gaussian curvature C from Ω . This numerical procedure gives results that are consistent with the expected asymptotic form: for example, in the case $a = 0$ one finds $1 - \Omega|K|^{1/3}/c \approx 10^{-3}$ when $|K| = 3$. The results are illustrated in Figure 3, which plots the curvature C versus K , for two values of the parameter a . The left-hand plot is for $a = 0$, and is rotationally-symmetric as expected. The right-hand one is for $a = 3$, and we see that the curvature is peaked at two points:

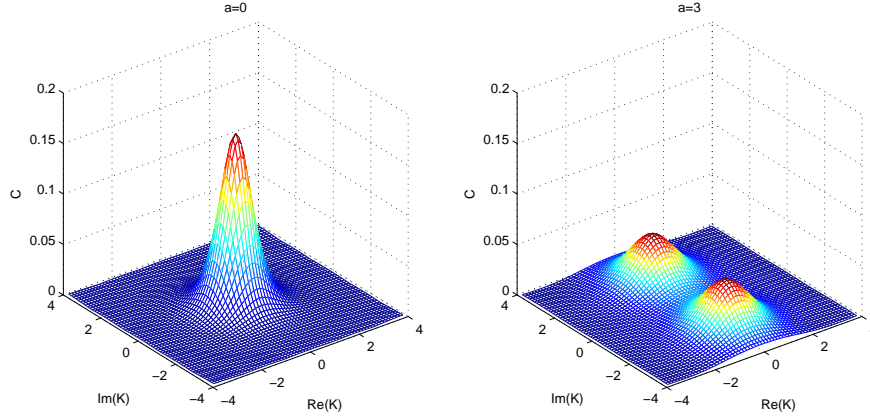


Figure 3: Gaussian curvature C of \mathcal{S}_+ , for $a = 0$ and $a = 3$.

these are the values $K = \pm 2(-a/3)^{3/2}$ for which the polynomial $H(z)$ has a double root, which means that two of the three lumps coincide.

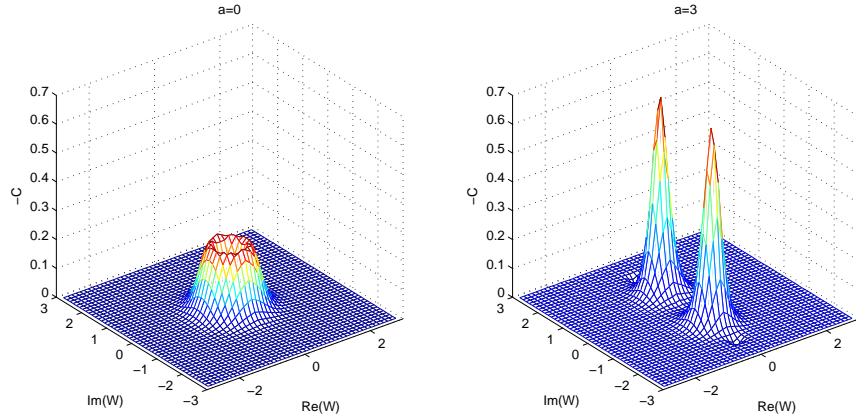


Figure 4: Curvature of \mathcal{S}_- , for $a = 0$ and $a = 3$.

Next we do the same thing for the other component \mathcal{S}_- . The surface \mathcal{S}_- is a triple cover of the K -plane, since any one of the three lumps can be the odd one out (antiparallel to the other two). Indeed, \mathcal{S}_- has a global complex coordinate W , where $W^3 + aW - K = 0$. Repeating the procedure described above leads to Figure 4. In this case, the Gaussian curvature C is (predominantly) negative, and $-C$ is plotted. Once again the $a = 0$ case is rotationally-symmetric, while in the $a = 3$ case the curvature has four peaks. There are two values of W , and hence K , for which a lump coincides with the antilump, leaving only

a single lump in \mathbb{R}^2 ; the curvature is negative at these values of W (in the right-hand plot of Figure 4, this is at $W = \pm i$). And there are two values of W for which the two parallel lumps coincide; the curvature is positive at those values ($W = \pm 2i$ in Figure 4).

The picture therefore is that the moduli space \mathcal{M}_a has an asymptotically-conical geometry; the vertices of this cone, which are located at the values of K for which $H(z) = \det \Phi$ has coincident zeros, are smoothed out, and the curvature of \mathcal{M}_a is concentrated at these loci.

6 Remarks

The natural energy functional for the Hitchin system, arising as a dimensional reduction of the Yang-Mills action in \mathbb{R}^4 , is

$$E = \int_{\mathbb{R}^2} (|D_j \Phi|^2 + |F|^2 + \frac{1}{4} |[\Phi, \Phi^*]|^2) dx^1 dx^2. \quad (16)$$

Solutions of the Hitchin equations (1) are critical points of (16). However, for the soliton-like solutions described in this paper, the integral (16) diverges, on account of the $|D_j \Phi|^2$ term. One could try to regularize the integral by subtracting a fixed function from the integrand, in other words one that does not depend on the moduli. But a calculation with the approximate solution quickly shows that this does not work: a moduli-dependent part remains which is at least logarithmically divergent. So it remains unclear as to whether the energy (16) can be regularized in a satisfactory way.

The information which determines a solution may also be understood in terms of spectral data, consisting of a bundle over a spectral curve, which is the zero-set of $F(z, t) = \det(\Phi(z) - t)$. In the $SU(2)$ case with $n = 3$ described above, the spectral curve is the standard elliptic curve $t^2 = z^3 + az - K$. The spectral data are preserved under the generalized Nahm transform. The details of this transform depend on the boundary conditions being imposed, and several different examples of Nahm transforms applied to the Hitchin equations have been studied [9, 20, 21, 22].

In this paper, we have only considered one particular moduli space, namely that of smooth $n = 3$ solutions with gauge group $SU(2)$. The same sort of methods can readily be applied to other cases, where n and the gauge group are different and where one allows singularities. One would expect there to be isometries between some of the resulting moduli spaces; a first step would be to classify these spaces up to isometry, using the Nahm transform, along the lines of what was done for doubly-periodic monopoles [23, 24].

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